

3-Rainbow index and forbidden subgraphs*

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Abstract

A tree in an edge-colored connected graph G is called a *rainbow tree* if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S -tree* if it connects S in G . A *k -rainbow coloring* of G is an edge-coloring of G having the property that for every set S of k vertices of G , there exists a rainbow S -tree in G . The minimum number of colors that are needed in a k -rainbow coloring of G is the *k -rainbow index* of G , denoted by $rx_k(G)$. The *Steiner distance* $d(S)$ of a set S of vertices of G is the minimum size of an S -tree T . The *k -Steiner diameter* $sdiam_k(G)$ of G is defined as the maximum Steiner distance of S among all sets S with k vertices of G . In this paper, we focus on the 3-rainbow index of graphs and find all finite families \mathcal{F} of connected graphs, for which there is a constant $C_{\mathcal{F}}$ such that, for every connected \mathcal{F} -free graph G , $rx_3(G) \leq sdiam_3(G) + C_{\mathcal{F}}$.

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1 Introduction

All graphs considered in this paper are simple, finite, undirected and connected. We follow the terminology and notation of Bondy and Murty [1] for those not defined here.

Let G be a nontrivial connected graph with an *edge-coloring* $c : E(G) \rightarrow \{1, 2, \dots, t\}$, $t \in \mathbb{N}$, where adjacent edges may be colored with the same color. A path in G is called a *rainbow path* if no two edges of the path are colored with the same color. The graph G is called *rainbow connected* if for any two distinct vertices of G , there is a rainbow

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path connecting them. For a connected graph G , the *rainbow connection number* of G , denoted by $rc(G)$, is defined as the minimum number of colors that are needed to make G rainbow connected. These concepts were first introduced by Chartrand et al. in [4] and have been well-studied since then. For further details, we refer the reader to a survey paper [8] and a book [9].

In [5], Chartrand et al. generalized the concept of rainbow path to rainbow tree. A tree in an edge-colored graph G is called a *rainbow tree* if no two edges of it are assigned the same color. For a vertex subset $S \subseteq V(G)$, a tree is called an *S -tree* if it connects S in G . Let G be a connected graph of order n . For a fixed integer k with $2 \leq k \leq n$, a *k -rainbow coloring* of G is an edge-coloring of G having the property that for every k -subset S of G , there exists a rainbow S -tree in G , and in this case, the graph G is called *k -rainbow connected*. The minimum number of colors that are needed in a k -rainbow coloring of G is the *k -rainbow index* of G , denoted by $rx_k(G)$. Clearly, $rx_2(G)$ is just the rainbow connection number $rc(G)$ of G . In the sequel, we assume that $k \geq 3$. It is easy to see that $rx_2(G) \leq rx_3(G) \leq \dots \leq rx_n(G)$. Recently, some results on the k -rainbow index have been published, especially on the 3-rainbow index. We refer to [3, 6] for more details.

The *Steiner distance* $d(S)$ of a set S of vertices in G is the minimum size of a tree in G containing S . Such a tree is called a *Steiner S -tree* or simply a *Steiner tree*. The *k -Steiner diameter* $sdiam_k(G)$ of G is defined as the maximum Steiner distance of S among all k -subsets S of G . Then the following observation is immediate.

Observation 1. [5] *For every connected graph G of order $n \geq 3$ and each integer k with $3 \leq k \leq n$,*

$$k - 1 \leq sdiam_k(G) \leq rx_k(G) \leq n - 1.$$

The authors of [5] showed that the k -rainbow index of trees can achieve the upper bound.

Proposition 1. [5] *Let T be a tree of order $n \geq 3$. For each integer k with $3 \leq k \leq n$,*

$$rx_k(T) = n - 1.$$

From above, we notice that for a fixed integer k with $k \geq 3$, the difference $rx_k(G) - sdiam_k(G)$ can be arbitrarily large. In fact, if G is a star $K_{1,n}$, then we have $rx_k(G) - sdiam_k(G) = n - k$.

They also determined the precise values for the k -rainbow index of the cycle C_n and the 3-rainbow index of the complete graph K_n .

Theorem 1. [5] *For integers k and n with $3 \leq k \leq n$,*

$$rx_k(C_n) = \begin{cases} n - 2 & \text{if } k = 3 \text{ and } n \geq 4 \\ n - 1 & \text{if } k = n = 3 \text{ or } 4 \leq k \leq n. \end{cases}$$

Theorem 2. [5]

$$rx_3(K_n) = \begin{cases} 2 & \text{if } 3 \leq n \leq 5 \\ 3 & \text{if } n \geq 6. \end{cases}$$

Let \mathcal{F} be a family of connected graphs. We say that a graph G is \mathcal{F} -free if G does not contain any induced subgraph isomorphic to a graph from \mathcal{F} . Specifically, for $\mathcal{F} = \{X\}$ we say that G is X -free, for $\mathcal{F} = \{X, Y\}$ we say that G is (X, Y) -free, and for $\mathcal{F} = \{X, Y, Z\}$ we say that G is (X, Y, Z) -free. The members of \mathcal{F} will be referred as *forbidden induced subgraphs* in this context. If $\mathcal{F} = \{X_1, X_2, \dots, X_k\}$, we also refer to the graphs X_1, X_2, \dots, X_k as a *forbidden k -tuple*, and for $|\mathcal{F}| = 2$ and 3 we also say *forbidden pair* and *forbidden triple*, respectively.

In [7], Holub et al. considered the question: For which families \mathcal{F} of connected graphs, a connected \mathcal{F} -free graph G satisfies $rc(G) \leq \text{diam}(G) + C_{\mathcal{F}}$, where $C_{\mathcal{F}}$ is a constant (depending on \mathcal{F}), and they gave a complete answer for $|\mathcal{F}| \in \{1, 2\}$ in the following two results (where N denotes the *net*, a graph obtained by attaching a pendant edge to each vertex of a triangle).

Theorem 3. [7] *Let X be a connected graph. Then there is a constant C_X such that every connected X -free graph G satisfies $rc(G) \leq \text{diam}(G) + C_X$, if and only if $X = P_3$.*

Theorem 4. [7] *Let X, Y be connected graphs such that $X, Y \neq P_3$. Then there is a constant C_{XY} such that every connected (X, Y) -free graph G satisfies $rc(G) \leq \text{diam}(G) + C_{XY}$, if and only if (up to symmetry) either $X = K_{1,r}$ ($r \geq 4$) and $Y = P_4$, or $X = K_{1,3}$ and Y is an induced subgraph of N .*

Let $k \geq 3$ be a positive integer. From Observation 1, we know that the k -rainbow index is lower bounded by the k -Steiner diameter. So we wonder an analogous question concerning the k -rainbow index of graphs. In this paper, we will consider the following question.

For which families \mathcal{F} of connected graphs, there is a constant $C_{\mathcal{F}}$ such that $rx_k(G) \leq \text{sdiam}_k(G) + C_{\mathcal{F}}$ if a connected graph G is \mathcal{F} -free ?

In general, it is very difficult to give answers to the above question, even if one considers the case $k = 4$. So, in this paper we pay our attention only on the case $k = 3$. In Sections 3, 4 and 5, we give complete answers for the 3-rainbow index when $|\mathcal{F}| = 1, 2$ and 3 , respectively. Finally, we give a complete characterization for an arbitrary finite family \mathcal{F} .

2 Preliminaries

In this section, we introduce some further terminology and notation that will be used in the sequel. Throughout the paper, \mathbb{N} denotes the set of all positive integers.

Let G be a graph. We use $V(G)$, $E(G)$, and $|G|$ to denote the vertex set, edge set, and the order of G , respectively. For $A \subseteq V(G)$, $|A|$ denotes the number of vertices in A , and $G[A]$ denotes the subgraph of G induced by the vertex set A . For two disjoint subsets X and Y of $V(G)$, we use $E[X, Y]$ to denote the set of edges of G between X and Y . For graphs X and G , we write $X \subseteq G$ if X is a subgraph of G , $X \stackrel{\text{IND}}{\subseteq} G$ if X

is an induced subgraph of G , and $X \cong G$ if X is isomorphic to G . In an edge-colored graph G , we use $c(uv)$ to denote the color assigned to an edge $uv \in E(G)$.

Let G be a connect graph. For $u, v \in V(G)$, a path in G from u to v will be referred as a (u, v) -path, and, whenever necessary, it will be considered with orientation from u to v . The *distance* between u and v in G , denoted by $d_G(u, v)$, is the length of a shortest (u, v) -path in G . The *eccentricity* of a vertex v is $\text{ecc}(v) := \max_{x \in V(G)} d_G(v, x)$. The *diameter* of G is $\text{diam}(G) := \max_{x \in V(G)} \text{ecc}(x)$, and the *radius* of G is $\text{rad}(G) := \min_{x \in V(G)} \text{ecc}(x)$. One can easily check that $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$. A vertex x is *central* in G if $\text{ecc}(x) = \text{rad}(G)$. Let $D \subseteq V(G)$ and $x \in V(G) \setminus D$. Then we call a path $P = v_0 v_1 \dots v_k$ is a v - D path if $v_0 = v$ and $V(P) \cap D = v_k$, and $d_G(v, D) := \min_{w \in D} d_G(v, w)$.

For a set $S \subseteq V(G)$ and $k \in \mathbb{N}$, we use $N_G^k(S)$ to denote the *neighborhood at distance k* of S , i.e., the set of all vertices of G at distance k from S . In the special case when $k = 1$, we simply write $N_G(S)$ for $N_G^1(S)$ and if $|S| = 1$ with $x \in S$, we write $N_G(x)$ for $N_G(\{x\})$. For a set $M \subseteq V(G)$, we set $N_M(S) = N_G(S) \cap M$ and $N_M(x) = N_G(x) \cap M$. Finally, we will also use the *closed neighborhood* of a vertex $x \in V(G)$ defined by $N_G^k[x] = (\cup_{i=1}^k N_G^i(x)) \cup \{x\}$.

A set $D \subseteq V(G)$ is called *dominating* if every vertex in $V(G) \setminus D$ has a neighbor in D . In addition, if $G[D]$ is connected, then we call D a *connected dominating set*. A *clique* of a graph G is a subset $Q \subseteq V(G)$ such that $G[Q]$ is complete. A clique is *maximum* if G has no clique Q' with $|Q'| > |Q|$. For a graph G , a subset $I \subseteq V(G)$ is called an *independent set* of G if no two vertices of I are adjacent in G . An independent set is *maximum* if G has no independent set I' with $|I'| > |I|$.

For two positive integers a and b , the *Ramsey number* $R(a, b)$ is the smallest integer n such that in any two-coloring of the edges of a complete graph on n vertices K_n by red and blue, either there is a red K_a (i.e., a complete subgraph on a vertices all of whose edges are colored red) or there is a blue K_b . Ramsey [10] showed that $R(a, b)$ is finite for any a and b .

Finally, we will use P_n to denote the path on n vertices. An edge is called a *pendant edge* if one of its end vertices has degree one.

3 Families with one forbidden subgraph

In this section, we characterize all possible connected graphs X such that every connected X -free graph G satisfies $rx_3(G) \leq \text{sdiam}_3(G) + C_X$, where C_X is a constant.

Theorem 5. *Let X be a connected graph. Then there is a constant C_X such that every connected X -free graph G satisfies $rx_3(G) \leq \text{sdiam}_3(G) + C_X$, if and only if $X = P_3$.*

Proof. We have that the graph G is a complete graph since G is P_3 -free. Then from Theorem 2, it follows that $rx_3(G) \leq 3 = \text{sdiam}_3(G) + 1$.

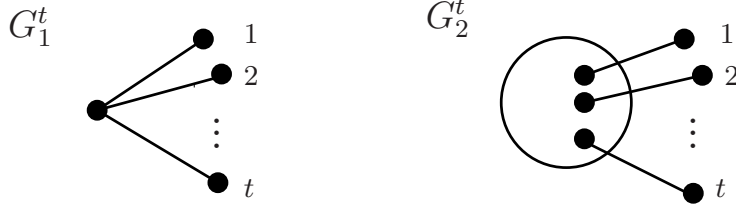


Figure 1: The graphs G_1^t and G_2^t .

Let t be an arbitrarily large integer, set $G_1^t = K_{1,t}$, and let G_2^t denote the graph obtained by attaching a pendant edge to each vertex of the complete graph K_t (see Figure 1). We also use K_t^h to denote G_2^t . Since $rx_3(G_1^t) = t$ but $sdiam_3(G_1^t) = 3$, X is an induced subgraph of G_1^t . Clearly, $rx_3(G_2^t) \geq t + 2$ but $sdiam_3(G_2^t) = 5$, and G_2^t is $K_{1,3}$ -free. Hence, $X = K_{1,2} = P_3$. The proof is thus complete. \square

4 Forbidden pairs

The following statement, which is the main result of this section, characterizes all possible forbidden pairs X, Y for which there is a constant C_{XY} such that $rx_3(G) \leq sdiam_3(G) + C_{XY}$ if G is (X, Y) -free. Since any P_3 -free graph is a complete graph, we exclude the case that one of X, Y is P_3 .

Theorem 6. *Let $X, Y \neq P_3$ be a pair of connected graphs. Then there is a constant C_{XY} such that every connected (X, Y) -free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$, if and only if (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.*

The proof of Theorem 6 will be divided into two parts. We prove the necessity in Proposition 2, and then we establish the sufficiency in Theorem 7.

Proposition 2. *Let $X, Y \neq P_3$ be a pair of connected graphs for which there is a constant C_{XY} such that every connected (X, Y) -free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C_{XY}$. Then, (up to symmetry) $X = K_{1,r}, r \geq 3$ and $Y = P_4$.*

Proof. Let t be an arbitrarily large integer, and set $G_3^t = C_t$. We will also use the graphs G_1^t and G_2^t shown in Figure 1.

Consider the graph G_1^t . Since $sdiam_3(G_1^t) = 3$ but $rx_3(G_1^t) = t$, we have, up to symmetry, $X = K_{1,r}, r \geq 3$. Then we consider the graphs G_2^t and G_3^t . It is easy to verify that $sdiam_3(G_2^t) = 5$ but $rx_3(G_2^t) \geq t + 2$, and $sdiam_3(G_3^t) = \lceil \frac{2}{3}t \rceil$ while $rx_3(G_3^t) \geq t - 2 \geq \frac{3}{2}(sdiam_3(G_3^t) - 1) - 2$, respectively. Clearly, G_2^t and G_3^t are both $K_{1,3}$ -free, so neither of them contains X , implying that both G_2^t and G_3^t contain Y . Since the maximum common induced subgraph of them is P_4 , we get that $Y = P_4$. This completes the proof. \square

Next, we can prove that the converse of Proposition 2 is true.

Theorem 7. Let G be a connected $(P_4, K_{1,r})$ -free graph for some $r \geq 3$. Then $rx_3(G) \leq sdiam_3(G) + r + 3$.

Proof. Let G be a connected $(P_4, K_{1,r})$ -free graph ($r \geq 3$). Then, $sdiam_3(G) \geq 2$. For simplicity, we set $V = V(G)$. Let $S \subseteq V$ be the maximum clique of G .

Claim 1: S is a dominating set.

Proof. Assume that there is a vertex y at distance 2 from S . Let yxu be a shortest path from y to S , where $u \in S$. Because S is the maximum clique, there is some $v \in S$ such that $vx \notin E(G)$. Thus the path $vuxy \cong P_4$, a contradiction. So S is a dominating set. \square

Let X be the maximum independent set of $G[V \setminus S]$ and $Y = V \setminus (S \cup X)$. Then for any vertex $y \in Y$, y is adjacent to some $x \in X$. Furthermore, for any independent set W of graph $G[Y]$, $|N_X(W)| \geq |W|$ since X is maximum.

Claim 2: There is a vertex $v \in S$ such that v is adjacent to all the vertices in X .

Proof. Suppose that the claim fails. Let u be the vertex of S with the largest number of neighbors in X . Set $X_1 = N_X(u)$, $X_2 = X \setminus X_1$. Then, $X_2 \neq \emptyset$ according to our assumption. Pick a vertex w in X_2 . Then, $uw \notin E(G)$. Let v be a neighbor of w in S . For any vertex z in X_1 , $G[w, v, u, z]$ can not be an induced P_4 , so vz must be an edge of G . Thus, $N_X(v) \supseteq N_X(u) \cup \{w\}$, contradicting the maximum of u . \square

Let z be the vertex in S which is adjacent to all the vertices of X . Set $X = \{x_1, x_2, \dots, x_\ell\}$. Then, $0 \leq \ell \leq r - 1$ since G is $K_{1,r}$ -free. Now we demonstrate a 3-rainbow coloring of G using at most $\ell + 6$ colors. Assign color i to the edge zx_i , and $i + 1$ to the edge x_iy where $1 \leq i \leq \ell$ and $y \in Y$. Color $E[S, Y]$ with color $\ell + 2$ and $E(G[Y])$ with color $\ell + 3$. Give a 3-rainbow coloring of $G[S]$ using colors from $\{\ell + 4, \ell + 5, \ell + 6\}$. And color the remaining edges arbitrarily (e.g., all of them with color 1). Next, we prove that this coloring is a 3-rainbow coloring of G .

Let $W = \{u, v, w\}$ be a 3-subset of V .

(i) $\{u, v, w\} \subseteq S \cup X$. There is a rainbow tree containing W .

(ii) $\{u, v\} \subseteq S \cup X, w \in Y$. We can find a rainbow tree containing an edge in $E[S, Y]$ that connects W .

(iii) $u \in S \cup X, \{v, w\} \subseteq Y$.

a) If $vw \in E(G)$, then there is a rainbow tree containing the edge vw that connects W .

b) If $vw \notin E(G)$, then we have $|N_X(\{v, w\})| \geq |\{v, w\}| = 2$. So there are two vertices x_i and x_j ($i \neq j$) in X adjacent to v and w , respectively. As $i + 1 \neq j + 1$, so either $i + 1 \neq c(zu)$ or $j + 1 \neq c(zu)$. Without loss of generality, we assume that $i + 1 \neq c(zu)$ and s is a neighbor of w in S . Then there is a rainbow tree containing the edges zu, uv, sw, sz if $u = x_i$ or the edges zu, zx_i, x_iv, sw, sz if $u \neq x_i$.

(iv) $\{u, v, w\} \subseteq Y$.

a) If $\{uv, vw, uw\} \cap E(G) \neq \emptyset$, for example, $uv \in E(G)$, then we have a rainbow tree containing the edges zx_i, x_iu, uv, sw, sz where x_i is a neighbor of u in X and s is a neighbor of w in S .

b) If $\{uv, vw, uw\} \cap E(G) = \emptyset$, then we have $|N_X\{u, v, w\}| \geq |\{u, v, w\}| = 3$, so we can find three distinct vertices x_i, x_j, x_k in X such that $\{x_iu, x_jv, x_kw\} \subseteq E(G)$. We may assume that $i < j < k$, so $k+1 \notin \{i, j, k, i+1, j+1\}$ and $k \neq i+1$. Then there is a rainbow tree containing the edges $zx_i, x_iu, zx_k, x_kw, sv, sz$ where s is a neighbor of v in S .

Thus the coloring is a 3-rainbow coloring of G using at most $\ell + 6 \leq r + 5 \leq \text{sdi}am_3(G) + r + 3$ colors. The proof is complete. \blacksquare

Combining Proposition 2 and Theorem 7, we can easily get Theorem 6.

Remark When the maximum independent set of $G[V \setminus S]$, X , satisfies $|X| = \ell \geq 4$, we just need $\ell + 5$ colors in the proof of Theorem 7: for the edges $x_\ell y$, we can color them with color 1 instead of color $\ell + 1$. It only matters when the case $\{u, v, w\} \subseteq Y$ and $\{uv, vw, uw\} \cap E(G) = \emptyset$ happens. Suppose $\{x_iu, x_jv, x_kw\} \subseteq E(G)$ and $i < j < k$. If $i \neq 1$ or $k \neq \ell$, it is the case in the proof above. So we turn to the case when $i = 1$ and $k = \ell$. If $j = 2$, then $j + 1 < 4 \leq \ell$ (that is why we need the condition $\ell \geq 4$). Thus, there is a rainbow tree containing the edges $zx_j, x_jv, zx_k, x_kw, su, sz$ where s is a neighbor of u in S . If $j \neq 2$, then there is a rainbow tree containing the edges $zx_i, x_iu, zx_j, x_jv, sw, sz$.

5 Forbidden triples

Now, we continue to consider more and obtain an analogous result which characterizes all forbidden triples \mathcal{F} for which there is a constant $C_{\mathcal{F}}$ such that G being \mathcal{F} -free implies $rx_3(G) \leq \text{sdi}am_3(G) + C_{\mathcal{F}}$. We exclude the cases which are covered by Theorems 5 and 6. We set:

$$\mathfrak{F}_1 = \{\{P_3\}\},$$

$$\mathfrak{F}_2 = \{\{K_{1,r}, P_4\} \mid r \geq 3\},$$

$$\mathfrak{F}_3 = \{\{K_{1,r}, Y, P_\ell\} \mid r \geq 3, Y \stackrel{\text{IND}}{\subseteq} K_s^h, s \geq 3, \ell > 4\}.$$

Theorem 8. *Let \mathcal{F} be a family of connected graphs with $|\mathcal{F}| = 3$ such that $\mathcal{F} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2$. Then there is a constant $C_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph G satisfies $rx_3(G) \leq \text{sdi}am_3(G) + C_{\mathcal{F}}$, if and only if $\mathcal{F} \in \mathfrak{F}_3$.*

First of all, we prove the necessity of the triples given by Theorem 8.

Proposition 3. *Let $X, Y, Z \neq P_3$ be connected graphs, $\{X, Y, Z\} \not\supseteq \mathcal{F}'$ for any $\mathcal{F}' \in \mathfrak{F}_2$, for which there is a constant C_{XYZ} such that every connected (X, Y) -free graph G*

satisfies $rx_3(G) \leq \text{sdi}_{\text{am}_3}(G) + C_{XYZ}$. Then, (up to symmetry) $X = K_{1,r}$ ($r \geq 3$), $Y \subseteq K_s^h$ ($s \geq 3$), and $Z = P_\ell$ ($\ell > 4$).

Proof. Let t be an arbitrarily large integer, and let G_1^t, G_2^t, G_3^t be the graphs defined in the proof of Proposition 2.

Firstly, we consider the graph G_1^t . Up to symmetry, we have $X = K_{1,r}$, $r \geq 3$ (for the case $r = 2$ is excluded by the assumptions). Secondly, we consider the graph G_2^t . The graph G_2^t does not contain X , since it is $K_{1,3}$ -free. Thus, up to symmetry, we have G_2^t contains Y , implying $Y \subseteq K_s^h$ for some $s \geq 3$ (for the case $s \leq 2$ is excluded by the assumptions). Finally, we consider the graphs G_3^t and G_3^{t+1} . Clearly, they are $(K_{1,3}, K_3^h)$ -free, so both of them contain neither X nor Y . Hence, we get that $Z = P_\ell$ for some $\ell > 4$ (for the case $\ell \leq 4$ is excluded by the assumptions).

This completes the proof. \square

It is easy to observe that if $X \subseteq^{\text{IND}} X'$, then every (X, Y, Z) -free graph is also (X', Y, Z) -free. Thus, when proving the sufficiency of Theorem 8, we will be always interested in *maximal triples* of forbidden subgraphs, i.e., triples X, Y, Z such that, if replacing one of X, Y, Z , say X , with a graph $X' \neq X$ such that $X \subseteq^{\text{IND}} X'$, then the statement under consideration is not true for (X', Y, Z) -free graphs.

For every vertex $c \in V(G)$ and $i \in \mathbb{N}$, we set $\alpha_i(G, c) = \max\{|M| \mid M \subseteq N_G^i[c], M \text{ is independent}\}$ and $\alpha_i^0(G, c) = \max\{|M^0| \mid M^0 \subseteq N_G^i(c), M^0 \text{ is independent}\}$.

Lemma 1. [2] *Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha(r, s, i)$ such that, for every connected $(K_{1,r}, K_s^h)$ -free graph G and for every $c \in V(G)$, $\alpha_i(G, c) < \alpha(r, s, i)$.*

We use the proof of Lemma 1 to get the following corollary concerning $\alpha_i^0(G, c)$ for each integer $i \geq 1$.

Corollary 1. *Let $r, s, i \in \mathbb{N}$. Then there is a constant $\alpha^0(r, s, i)$ such that, for every connected $(K_{1,r}, K_s^h)$ -free graph G and for every $c \in V(G)$, $\alpha_i^0(G, c) < \alpha^0(r, s, i)$.*

Proof. For the sake of completeness, here we give a brief proof concentrating on the upper bound of $\alpha_i^0(G, c)$. We prove the corollary by induction on i .

For $i = 1$, we have $\alpha^0(r, s, 1) = r$, for otherwise G contains a $K_{1,r}$ as an induced subgraph.

Let, to the contrary, i be the smallest integer for which $\alpha^0(r, s, i)$ does not exist (i.e., $\alpha_i^0(G, c)$ can be arbitrarily large), choose a graph G and a vertex $c \in V(G)$ such that $\alpha_i^0(G, c) \geq (r-2)R(s(2r-3), \alpha^0(r, s, i-1))$, and let $M^0 = \{x_1^0, \dots, x_k^0\} \subseteq N_G^i(c)$ be an independent set in G of size $\alpha_i^0(G, c)$. Obviously, $k \geq (r-2)R(s(2r-3), \alpha^0(r, s, i-1))$. Let Q_j be a shortest (x_j^0, c) -path in G , $j = 1, \dots, k$. We denote $M^1 \subseteq N_G^{i-1}(c)$ the set of all successors of the vertices from M^0 on Q_j , $j = 1, \dots, k$, and x_j^1 the successor of x_j^0 on Q_j (note that some distinct vertices in M^0 can have a common successor in M^1). Every vertex in M^1 has at most $r-2$ neighbors in M^0 since G is $K_{1,r}$ -free.

Thus, $|M^1| \geq \frac{k}{r-2} \geq R(s(2r-3), \alpha^0(r, s, i-1))$. By the induction assumption and the definition of Ramsey number, $G[M^1]$ contains a complete subgraph $K_{s(2r-3)}$. Choose the notation such that $V(K_{s(2r-3)}) = \{x_1^1, \dots, x_{s(2r-3)}^1\}$, and set $\widetilde{M^0} = N_{M^0}(K_{s(2r-3)})$. Using a matching between $K_{s(2r-3)}$ and $\widetilde{M^0}$, we can find in G an induced K_s^h with vertices of degree 1 in $\widetilde{M^0}$, a contradiction. For more details about finding the K_s^h , we refer the reader to [2]. \square

Armed with Corollary 1, we can get the following important theorem.

Theorem 9. *Let $r \geq 3, s \geq 3$, and $\ell > 4$ be fixed integers. Then there is a constant $C(r, s, \ell)$ such that every connected $(K_{1,r}, K_s^h, P_\ell)$ -free graph G satisfies $rx_3(G) \leq sdiam_3(G) + C(r, s, \ell)$.*

Proof. We have $diam(G) \leq \ell - 2$ since G is P_ℓ -free. Let c be a central vertex of G , i.e., $ecc(c) = rad(G) \leq diam(G) \leq \ell - 2$. And we set $S_i = \cup_{j=1}^i N_G^j[c]$ for an integer $i \geq 1$.

Claim: $rx_3(G[S_i \cup N_G^{i+1}(c)]) \leq rx_3(G[S_i]) + \alpha_{i+1}^0(G, c) + 3$

Proof. Let $X = \{x_1, x_2, \dots, x_{\alpha_{i+1}^0(G, c)}\}$ be the maximum independent set of $N_G^{i+1}(c)$ and $Y = N_G^{i+1}(c) \setminus X$. Then for any vertex $y \in Y$, y is adjacent to some $x \in X$ and $s \in S_i$. Further more, for any independent set W of graph $G[Y]$, we have $|N_X(W)| \geq |W|$ since X is maximum.

Now we demonstrate a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$ using at most $k + \alpha_{i+1}^0(G, c) + 3$ colors, where $k = rx_3(G[S_i])$. We color the edges of $G[S_i]$ using colors $1, 2, \dots, k$. Color $E[S_i, Y]$ with color $k + 1$ and $E(G[Y])$ with color $k + 2$. And assign color $j + k + 2$ to the edges $E[\{x_j\}, S_i]$, and $j + k + 3$ to the edges $E[\{x_j\}, Y]$ where $1 \leq j \leq \alpha_{i+1}^0(G, c)$. With the same argument as the proof of Theorem 7, we can prove that this coloring is a 3-rainbow coloring of $G[S_i \cup N_G^{i+1}(c)]$. \square

From the proof of Corollary 1, it follows that $\alpha_1^0(G, c) \leq r - 1$ and $\alpha_i^0(G, c) \leq (r-2)R(s(2r-3), \alpha^0(r, s, i-1)) - 1$ for each integer $i \geq 2$. Let $\mathcal{R}(r, s) = \sum_{i=2}^{ecc(c)} R(s(2r-3), \alpha^0(r, s, i-1))$. Recall that $ecc(c) \leq \ell - 2$. Repeated application of Claim gives the following:

$$\begin{aligned} rx_3(G) &\leq rx_3(G[N_G^{ecc(c)-1}[c]]) + \alpha_{ecc(c)}^0(G, c) + 3 \\ &\leq \dots \\ &\leq rx_3(c) + \alpha_1^0(G, c) + \dots + \alpha_{ecc(c)}^0(G, c) + 3ecc(c) \\ &\leq 0 + r + (r-2)\mathcal{R}(r, s) + 2(\ell-2) \\ &\leq sdiam_3(G) + (r-2)(\mathcal{R}(r, s) + 1) + 2(\ell-1). \end{aligned}$$

Thus, we complete our proof. \blacksquare

Remark The same as the remark in Section 4: for $i \geq 1$, every time $\alpha_{i+1}^0(G, c) \geq 4$ happens, we can save one color in the Claim of Theorem 9.

6 Forbidden k -tuples for any $k \in \mathbb{N}$

Let $\mathcal{F} = \{X_1, X_2, X_3, \dots, X_k\}$ be a finite family of connected graphs with $k \geq 4$ for which there is a constant $k_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph satisfies $rx_3(G) \leq \text{sdi}_{\text{am}_3}(G) + C_{\mathcal{F}}$. Let t be an arbitrarily large integer, and let G_1^t, G_2^t and G_3^t be defined in Proposition 2. For the graph G_1^t , Up to symmetry, we suppose that $X_1 = K_r, r \geq 3$ (for the case $r = 2$ has been discussed in Section 3). Then, we consider the graphs G_2^t and G_3^t . Notice that G_2^t and G_3^t are both $K_{1,3}$ -free, so neither of them contains X_1 , implying that G_2^t or G_3^t contains X_i , where $i \neq 1$. We may assume that X_2 is an induced subgraph of G_2^t . If G_3^t contains X_2 , then $X_2 = P_4$, which is just the case in Section 4. So we turn to the case that G_3^t contains X_i for some $i > 2$. Now consider the graphs $G_3^t, G_3^{t+1}, G_3^{t+2}, \dots, G_3^{t+k}$, each of which contains at least one of X_3, X_4, \dots, X_k as an induced subgraph due to the analysis above. So it is forced that at least one of these $X_i (i \geq 3)$ is isomorphic to P_l for some $l \geq 5$, which goes back to the case in Section 5. Thus, the conclusion comes out.

Theorem 10. *Let \mathcal{F} be a finite family of connected graphs. Then there is a constant $C_{\mathcal{F}}$ such that every connected \mathcal{F} -free graph satisfies $rx_3(G) \leq \text{sdi}_{\text{am}_3}(G) + C_{\mathcal{F}}$, if and only if \mathcal{F} contains a subfamily $\mathcal{F}' \in \mathfrak{F}_1 \cup \mathfrak{F}_2 \cup \mathfrak{F}_3$.*

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